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Preliminary remarks of polynomial approximations for computers

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ABSTRACT

Polynomial approximations for 33 common mathematical functions (Trigonometric, Inverse Trigonometric, Exponential, Logarithmic, Inverse Hyperbolic, Gamma, Error, Exponential Integral and Bessel Functions) were obtained from truncations of the Chebyshev series for each function. Several approximations with a truncation error ranging from about $\pm 5 \times 10^{-3}$ to $\pm 5 \times 10^{-21}$ were given for each of these functions. These polynomial approxi mations (also available on IBM punched cards) may be obtained by writing to the above Computing Center.

1. *Introduction*

In the manipulation of mathematical functions by computers, polynomial or rational approximations are often preferred. Polynomial approximations over a range of truncation error from about $\pm 5 \times 10^{-3}$ to $\pm 5 \times 10^{-21}$ for most of the mentioned functions were given in 33 Tables.

The chief application for these approximations is to facilitate the writing of variable field length subroutines. Since the format of all the approximations is the same, the selection of a different set of constants will provide the evaluation of a function with different specified maximum errors. It is also possible, by a suitable adjustment, to evaluate different mathematical functions by the same subroutine using approximations in polynomial form.

2. *Method of Obtaining Approximations*

Clenshaw (1961) published 33 mathematical functions in terms of Chebyshev series to 20 figures (except 2 cases). A program was written for the IBM 1620 computer to truncate the Chebyshev series and to expand the remaining series in powers of x. All internal calculations used 20 decimal places.

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Coefficients of Chebyshev polynomials were evaluated using the recurrence formula.

$$T_n(x) = 2 \times T_{n-1}(x) - T_{n-2}(x)$$

with $$T_0(x) = 1, \; T_1(x) = x$$

and $$T_n^*(x^2) = T_{2n}(x).$$

The coefficients of Chebyshev polynomials, b_j, were checked by

$$\sum_j^n b_j = 1,$$

and were also compared with the published values up to $T_{20}^*(x)$ (N.B.S., 1952).

The coefficients for the polynomial approximations, C_i, were checked by desk calculations for the first and second sets for each function and all sets were checked by

$$\sum_i^n C_i = \sum_j^n A_j,$$

where the right hand side was given by Clenshaw.

3. *Contents of Tables of Polynomial Approximations*

The following notation was used

« 1) Sin $(\pi x/2)$, $(-1, x, 1)$, $(8\colon 3, M, 10)$, $(1.4 \times 10^{-4}, \triangle, 5 \times 10^{-21})$ »

means that Table 1 consists of approximations for Sin $(\pi x/2)$, the range being $-1 \leq x \leq 1$, and there are 8 sets of approximations from 3-term to 10-term polynomials with an absolute value of truncation error, $\triangle$, ranging from $\leq 1.4 \times 10^{-4}$ to $\leq 5 \times 10^{-21}$. All $\triangle$ may be out by 5×10^{-21}. The notation of functions is that of Clenshaw.

I. Trigonometric Functions

1) Sin $(\pi x/2)$, $(-1, x, 1)$, $(8\colon 3, M, 10)$, $(1.4 \times 10^{-4}, \triangle, 5 \times 10^{-21})$
2) Cos $(\pi x/2)$, $(-1, x, 1)$, $(9\colon 3, M, 11)$, $(6.0 \times 10^{-4}, \triangle, 5 \times 10^{-21})$
3) Sin (πx), $(-1, x, 1)$, $(10\colon 4, M, 13)$, $(5.2 \times 10^{-4}, \triangle, 5 \times 10^{-21})$
4) Cos (πx), $(-1, x, 1)$, $(10\colon 4, M, 13)$, $(1.4 \times 10^{-3}, \triangle, 5 \times 10^{-21})$
5) Tan $(\pi x/4)$, $(-1, x, 1)$, $(16\colon 3, M, 18)$, $(5.8 \times 10^{-4}, \triangle, 5 \times 10^{-21})$
6) Cot $(\pi x/2)$, $(-1, x, 1)$, $(17\colon 3, M, 19)$, $(1.2 \times 10^{-3}, \triangle, 5 \times 10^{-21})$

II. Inverse Trigonometric Functions

7) $\mathrm{Sin}^{-1}\ x$, $(-1/\sqrt{2}, x, 1/\sqrt{2})$, $(23:2, M, 24)$, $(3.2\times10^{-3}, \triangle, 5\times10^{-21})$

8) $\mathrm{Tan}^{-1}\ x$, $(-1, x, 1)$, $(23:3, M, 25)$, $(1.6\times10^{-3}, \triangle, 5\times10^{-21})$

III. Exponential Functions

9) e^x, $(-1, x, 1)$, $(14:5, M, 18)$, $(5.9\times10^{-4}, \triangle, 5\times10^{-21})$

10) 2^{-x}, $(0, x, 1)$, $(12:3, M, 14)$, $(1.3\times10^{-3}, \triangle, 5\times10^{-21})$

IV. Logarithmic Function

11) $1n\ (1+x)$, $(0, x, 1)$, $(24:3, M, 26)$, $(3.9\times10^{-3}, \triangle, 5\times10^{-21})$

V. Inverse Hyperbolic Functions

12) $\mathrm{Sin}h^{-1}\ x$, $(-1, x, 1)$, $(22:3, M, 24)$, $(5.6\times10^{-4}, \triangle, 5\times10^{-21})$

13) $\mathrm{Sin}h^{-1}\ x$, $(1, x, \infty)$, $(22:3, M, 24)$, $(6.8\times10^{-4}, \triangle, 5\times10^{-21})$

VI. Gamma Functions

14) Gamma $(1+x)$, $(0, x, 1)$, $(24:4, M, 27)$, $(1.6\times10^{-3}, \triangle, 5\times10^{-21})$

15) 1/Gamma $(1+x)$, $(0, x, 1)$, $(15:4, M, 18)$, $(5.0\times10^{-4}, \triangle, 5\times10^{-21})$

VII. Error Functions

16) erf (x), $(-4, x, 4)$, $(26:8, M, 33)$, $(4.5\times10^{-3}, \triangle, 5\times10^{-21})$

17) $y\ (x)$, $(4, x, \infty)$, $(17:2, M, 18)$, $(3.1\times10^{-4}, \triangle, 5\times10^{-21})$

VIII. Exponential Integral

18) $-E_i\ (x)$, $(-4, x, 4)$, $(20:8, M, 27)$, $(2.8\times10^{-3}, \triangle, 5\times10^{-21})$

19) $E_i\ (x)$, $(4, x, \infty)$, $(31:2, M, 32)$, $(5.0\times10^{-3}, \triangle, 5\times10^{-21})$

IX. Bessel Functions

20) $J_0\ (x)$, $(-8, x, 8)$, $(12:7, M, 18)$, $(4.9\times10^{-4}, \triangle, 5\times10^{-21})$

21) $Y_0\ (x)$, $(0, x, 8)$, $(13:7, M, 19)$, $(8.1\times10^{-4}, \triangle, 5\times10^{-21})$

22) $J_1\ (x)$, $(-8, x, 8)$, $(13:6, M, 18)$, $(3.5\times10^{-3}, \triangle, 5\times10^{-21})$

23) $Y_1\ (x)$, $(0, x, 8)$, $(12:7, M, 18)$, $(4.7\times10^{-4}, \triangle, 5\times10^{-21})$

24) $P_0\ (x)$, $(8, x, \infty)$, $(17:2, M, 18)$, $(3.1\times10^{-6}, \triangle, 5\times10^{-21})$

25) $Q_0\ (x)$, $(8, x, \infty)$, $(17:2, M, 18)$, $(7.6\times10^{-7}, \triangle, 5\times10^{-21})$

26) $P_1(x)$, $(8, x, \infty)$, $(17:2, M, 18)$, $(4.1 \times 10^{-6}, \Delta, 5 \times 10^{-21})$
27) $Q_1(x)$, $(8, x, \infty)$, $(17:2, M, 18)$, $(9.4 \times 10^{-7}, \Delta, 5 \times 10^{-21})$
28) $I_0(x)$, $(-8, x, 8)$, $(13:7, M, 19)$, $(3.6 \times 10^{-3}, \Delta, 5 \times 10^{-21})$
29) $K_0(x)$, $(0, x, 8)$, $(13:7, M, 19)$, $(2.7 \times 10^{-3}, \Delta, 5 \times 10^{-21})$
30) $I_1(x)$, $(-8, x, 8)$, $(12:7, M, 18)$, $(1.6 \times 10^{-3}, \Delta, 5 \times 10^{-21})$
31) $K_1(x)$, $(0, x, 8)$, $(12:7, M, 18)$, $(1.3 \times 10^{-3}, \Delta, 5 \times 10^{-21})$
32) $F_0(x)$, $(-1, 8/x, 1)$, $(25:2, M, 26)$, $(2.4 \times 10^{-4}, \Delta, 5 \times 10^{-15})$
33) $F_1(x)$, $(-1, 8/x, 1)$, $(25:2, M, 26)$, $(4.0 \times 10^{-4}, \Delta, 5 \times 10^{-15})$.

4. *Contents of Tables of Polynomial Approximations in Punched Cards*

All tables of polynomial approximations were punched on IBM cards as follows:

column 74-75: Table Number, No.
column 76-77: Number of terms in the polynomial, M.
column 78-80: The power of the variable, i.

Tables 16, 18, 28, 29, 30, and 31 were punched with

column 1-30: Coefficient of the polynomial approximation, C_i, with leading flag at Column 1, « – » sign as flag at column 30 and decimal point between columns 10 and 11.

The remaining Tables were punched with

column 1-25: with leading flag at column 1, « – » sign as flag at column 25 and decimal point between columns 5 and 6.

5. *Information for Further Reference*

Copies of 33 Tables of Polynomial Approximations and their punched cards may be obtained by writing to: Computing Center, University of Alberta, Edmonton, Alberta, Canada. In addition copies of Tables of Chebyshev Polynomials and their punched cards up to $T_{130}(x)$ are available upon request.

The author welcomes comments and suggestions which should be sent to author's present address.

REFERENCES

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